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A Note on λ_2 and λ_n of a Graph

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Abstract. Using the eigenvalues and eigenvectors of a graph G, it was established the upper bound for the second eigenvalue λ_2 and the least eigenvalue λ_n [1]. In this work using only the eigenvalues of G we obtain the upper bound for λ_2 and λ_n.

Let G be a graph of order n and let A be its ordinary adjacency matrix. The spectrum of G is the set of its eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. We say that λ_i is the i-th eigenvalue of G $(i = 1, 2, \ldots, n)$. In particular, λ_2 is called the second eigenvalue while λ_n is called the least eigenvalue. Using the eigenvalues and eigenvectors of G it was obtained the upper bound for λ_2 and λ_n (see [1], p. 222). In this paper we obtain the upper bound for λ_2 and λ_n using only the eigenvalues of G.

Theorem 1 ([3])**.** *Let G be a graph of order n, and let $\{\lambda_i\}$ and $\{\overline{\lambda}_i\}$ be the corresponding eigenvalues of G and its complement $\overline{G}$, respectively. Then:*

$$\lambda_i + \overline{\lambda}_{n+1-i} + 1 \geq 0 \qquad (i = 1, 2, \ldots, n), \tag{1}$$

$$\lambda_{i+1} + \overline{\lambda}_{n+1-i} + 1 \leq 0 \qquad (i = 1, 2, \ldots, n-1). \tag{2}$$

Theorem 2. *If G is a graph with n vertices then $\lambda_2 \leq \dfrac{n-2}{2}$.*

Proof. Assume, on the contrary, that there exists a graph G of order n with $\lambda_2 > \dfrac{n-2}{2}$. Let $S = \dfrac{1}{n-2}\sum_{i=3}^{n} |\lambda_i|$. Then we can see that

$$(n-2)S^2 \leq \sum_{i=3}^{n} |\lambda_i|^2. \tag{3}$$

Since

$$\sum_{\lambda_i<0} |\lambda_i| = \sum_{\lambda_i>0} |\lambda_i|$$

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and

$$\lambda_1 \geq \lambda_2 > \frac{n-2}{2}$$

we have

$$S \geq \frac{1}{n-2} \sum_{\lambda_i < 0} |\lambda_i| = \frac{1}{n-2} \sum_{\lambda_i > 0} |\lambda_i| \geq \frac{1}{n-2}(\lambda_1 + \lambda_2) > 1. \tag{4}$$

From relations (3) and (4) we have

$$\sum_{i=3}^{n} |\lambda_i|^2 \geq (n-2) \cdot S^2 > n-2. \tag{5}$$

Let m and $\overline{m}$ be the numbers of edges of the graphs G and $\overline{G}$, respectively. Then from (5) and using relations (1) and (2), we find that:

$$n^2 - n = 2m + 2\overline{m} = \sum_{i=1}^{n} \lambda_i^2 + \sum_{i=1}^{n} \overline{\lambda}_i^2 \geq \lambda_1^2 + \lambda_2^2 + \sum_{i=3}^{n} \lambda_i^2 + \overline{\lambda}_1^2 + \overline{\lambda}_n^2 >$$
$$> 2\left(\frac{n-2}{2}\right)^2 + (n-2) + 2\left(\frac{n}{2}\right)^2 = n^2 - n,$$

which is a contradiction. □

Corollary 1. *If* $\lambda_1 \in \left(\frac{n-2}{2}, n-1\right]$ *then* λ_1 *is the simple eigenvalue.*

Further, let G be non-regular graph of order n. We know that $\lambda_1(G) = d(G) + \Delta(G)$, where $d(G)$ denotes the mean value of the vertex degrees of G and $\Delta(G) > 0$. In view of this,

$$\lambda_1(G) + \lambda_1(\overline{G}) = n - 1 + \Delta(G) + \Delta(\overline{G}).$$

The proof of the next result is based on a property of the so-called canonical graphs [2].

We say that two vertices $x, y \in V(G)$ are equivalent in G and write $x \sim y$ if x is non-adjacent to y, and x and y have exactly the same neighbors in G. Relation $\sim$ is an equivalence relation on the vertex set $V(G)$. The corresponding quotient graph is denoted by $\widetilde{G}$, and is called the canonical graph of G.

We say that G is canonical if $|G| = |\widetilde{G}|$, that is if G has no two equivalent vertices. Let $\widetilde{G}$ be the canonical graph of G, $|\widetilde{G}| = k$, and $N_1, N_2, \ldots, N_k$ be the corresponding sets of equivalent vertices in G. Then we denote $G = \widetilde{G}(N_1, N_2, \ldots, N_k)$, or simply $G = \widetilde{G}(n_1, n_2, \ldots, n_k)$, where $|N_i| = n_i (i = 1, 2, \ldots, k)$.

In the case that $|N_i| = m$ for $i = 1, 2, \ldots, k)$, the corresponding graph $\widetilde{G}(m, m, \ldots, m)$ is denoted by G_{mk}. With this notation in [2] was proved the following result:

Proposition 1. *Let* $\widetilde{G}$ *be a canonical graph of order* k, *and let* $\{\lambda_i\}$ *and* $\{\overline{\lambda}_i\}$ *be the corresponding eigenvalues of* $\widetilde{G}$ *and its complement* $\overline{\widetilde{G}}$, *respectively. Then*

(1°) $H_{G_{mk}}(t) = mH_{\widetilde{G}}(mt)$;
(2°) $\sigma(G_{mk}) = \{m\lambda_i \mid i = 1, 2, \ldots, k\} \bigcup \{\underbrace{0, 0, \ldots, 0}_{n-k}\}$;
(3°) $\sigma(\overline{G}_{mk}) = \{m\overline{\lambda}_i + m - 1 \mid i = 1, 2, \ldots, k\} \bigcup \{\underbrace{-1, -1, \ldots, -1}_{n-k}\}$,

where $H_G(t)$ is the generating function of the numbers of walks in the graph G.

Lemma 1. *Let G be non-regular graph of order n. Then for every $M > 0$ there exists a graph $G^* \supseteq G$ of order $n^* = m \cdot n$ such that*

$$\lambda_1(G^*) = d(G^*) + \Delta(G^*) \geq d(G^*) + M,$$

and

$$\lambda_1(G^*) + \lambda_1(\overline{G}^*) \geq n^* - 1 + M.$$

Proof. Let $\{\lambda_i \mid i = 1, 2, \ldots, n\}$ be the eigenvalues of G. Then $\{m\lambda_i \mid i = 1, 2, \ldots, n\} \bigcup \{\underbrace{0, 0, \cdots, 0}_{n^*-n}\}$ are the eigenvalues of G^*. We now obtain the proof using the fact that $\lambda_1(G^*) = m \cdot \lambda_1(G)$ and $d(G^*) = m \cdot d(G)$. □

Theorem 3. *If G is a non-regular graph with n vertices then $|\lambda_n| < \dfrac{n}{2}$.*

Proof. We can suppose, on the contrary, that there exists a non-regular graph G of order n whit $|\lambda_n| \geq \dfrac{n}{2}$. Let m be the least integer such that $\Delta(G^*) \geq 2$. Then, $|\lambda_n(G^*)| \geq \dfrac{n^*}{2}$, where n^* is the order of G^*. For $k \in N$ we consider $G_k^* = \bigcup\limits_{i=1}^{k} G^*$. Then $|G_k^*| = k \cdot n^* = k \cdot m \cdot n$, and its eigenvalues are $m \cdot \lambda_1 \geq m \cdot \lambda_2 \geq \cdots \geq m \cdot \lambda_n$ of multiplicity k, while 0 is the eigenvalue of G^* of the multiplicity $k \cdot n^* - k \cdot n$.

Now, we have

$$(6)\quad n^{*2} \cdot k^2 - n^* \cdot k = \sum_{i=1}^{n^*k} \lambda_i^{*2} + \sum_{i=1}^{n^* \cdot k} \overline{\lambda}_i^{*2} = k\lambda_1^{*2} + \cdots + k\lambda_n^{*2} + k\overline{\lambda}_1^{*2} + \cdots + k\overline{\lambda}_n^{*2}.$$

Using relations (1), (2) and (6) we have

$$(7)\quad n^{*2} \cdot k^2 - n^* \cdot k \geq k\lambda_1^{*2} + k(\frac{n^*}{2})^2 + \overline{\lambda}_1^{*2} + (k-1)(\frac{n^*}{2} - 1)^2 + (k-1)(\lambda_1 + 1)^2.$$

Using (7) by an easy calculation we find that

$$n^{*2} \cdot k^2 - n^* \cdot k \geq k \cdot \frac{n^{*2}}{2} - \frac{n^{*2}}{4} + 2(k-1) + f(\lambda_1, \overline{\lambda}_1),$$

where $f(\lambda_1, \overline{\lambda}_1) \equiv (2k-1)\lambda_1^{*2} + \overline{\lambda}_1^{*2}$.

Next, we obtain that

$$\min f(\lambda_1, \overline{\lambda}_1) = n^{*2} \cdot k^2 - \frac{k \cdot n^{*2}}{2} + \frac{(2k-1)(\Delta_* - 1)^2}{2k} + n^*(2k-1)(\Delta_* - 1).$$

Since $\lambda_1^* + \overline{\lambda}_1^* = n^*k - 1 + \Delta_*$, where $\Delta_* = \Delta(G_k^*) + \Delta(\overline{G}_k^*) \geq \Delta(G_k^*) \geq 2$, we get

$$n^{*2} \cdot k^2 - n^* \cdot k \geq n^{*2}k^2 - \frac{n^{*2}}{4} + (2k-1) + n^* \cdot (2k-1),$$

a contradiction. □

Corollary 2. *For every regular graph G, $|\lambda_n| \leq \dfrac{n}{2}$.*

Proof. We can assume, on the contrary case, that there exists a regular graph G of order n, with $|\lambda_n| > \dfrac{n}{2}$. Let $|\lambda_n| = \dfrac{n}{2} + \varepsilon \quad (\varepsilon > 0)$. Then there exists a graph G^* of order $n^* = m \cdot n$ so that

$$|\lambda_n(G^*)| = m \cdot |\lambda_n(G)| > \frac{n^*}{2} + 1. \tag{8}$$

Let $G_* = G^* \bigcup K_1$, where K_1 is the graph with one isolated vertex. Since G_* is non-regular and according to theorem 3,

$$|\lambda_n(G_*)| < \frac{n(G_*)}{2} = \frac{n^*+1}{2},$$

we get a contradiction to relation (9). □

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